Exercise 2.2.14

(Energy-balance model for Earth's temperature) Climate scientists use a hierarchy of mathematical models, ranging from very detailed to very simplified. At one end of the spectrum are comprehensive models containing billions of variables about the state of the atmosphere and oceans, also taking account of sea ice, terrestrial vegetation, ecosystems, biogeochemical cycles (such as the carbon cycle), and atmospheric chemistry. At the other extreme are simple conceptual models. These are a lot easier to understand, yet they can still provide valuable insights.

One simple model summarizes the climate in a single variable: the mean surface temperature of the Earth, averaged over the entire globe. The model ignores differences in the atmosphere's composition, as well as differences among continents and oceans, topography, and all other local features. Its dynamics are governed by the following equation:

$$\begin{aligned} C\frac{dT}{dt} &= E_{\text{in}} - E_{\text{out}} \\ &= (1-\alpha)Q - \epsilon\sigma T^4. \end{aligned}$$

Here C is the heat capacity of the Earth, and T is the mean surface temperature (measured in degrees Kelvin) at time t. This is called an *energy-balance model* because it assumes the mean temperature changes in response to the amount of energy reaching the Earth from the Sun (E_{in}, proportional to the solar irradiance parameter Q) minus the energy emitted back out into the stratosphere (E_{out}, assumed to follow the Stefan-Boltzmann law). The parameter $0 < \alpha < 1$ is called the *albedo*; it's the fraction of incoming solar radiation that the Earth reflects. Its numerical value is different for ice, water, and land, but those details are averaged out in this simplified model. Likewise, $0 < \epsilon < 1$ accounts in an averaged way for the fact that some of the outgoing radiative energy does not make it to outer space; instead, a fraction of it gets absorbed by greenhouse effects in the atmosphere.

a) Find the model's prediction for Earth's mean surface temperature T^* in steady state, expressed in terms of the other parameters.

b) Show graphically that T^* is a *stable* fixed point.

c) Sketch T^* versus Q.

d) Estimate the numerical value of T^* in degrees Kelvin. (You may assume $Q = 342$ Watts/(meter)^2, $\alpha \approx 0.3$, $\sigma = 5.67 \times 10^{-8}$ Watts/(meter)$^2 K^4$, and $\epsilon \approx 0.62$.)

This analysis is from Kaper and Engler (2013).

[Write K as (Kelvin) to be consistent.]

Solution

Part a)

Set $dT/dt = 0$ to get an equation for the steady-state temperature T^*.

$$C\frac{dT}{dt} = (1-\alpha)Q - \epsilon\sigma T^4$$

$$0 = (1-\alpha)Q - \epsilon\sigma T^{*4}$$

Solve for T^*.

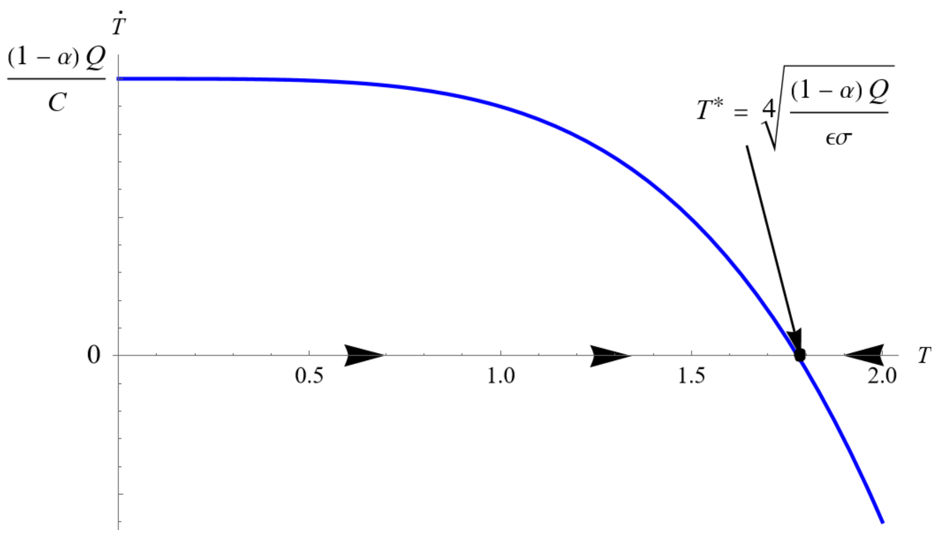
$$\epsilon\sigma T^{*4} = (1-\alpha)Q$$

$$T^{*4} = \frac{(1-\alpha)Q}{\epsilon\sigma}$$

$$\boxed{T^* = \sqrt[4]{\frac{(1-\alpha)Q}{\epsilon\sigma}}}$$

Part b)

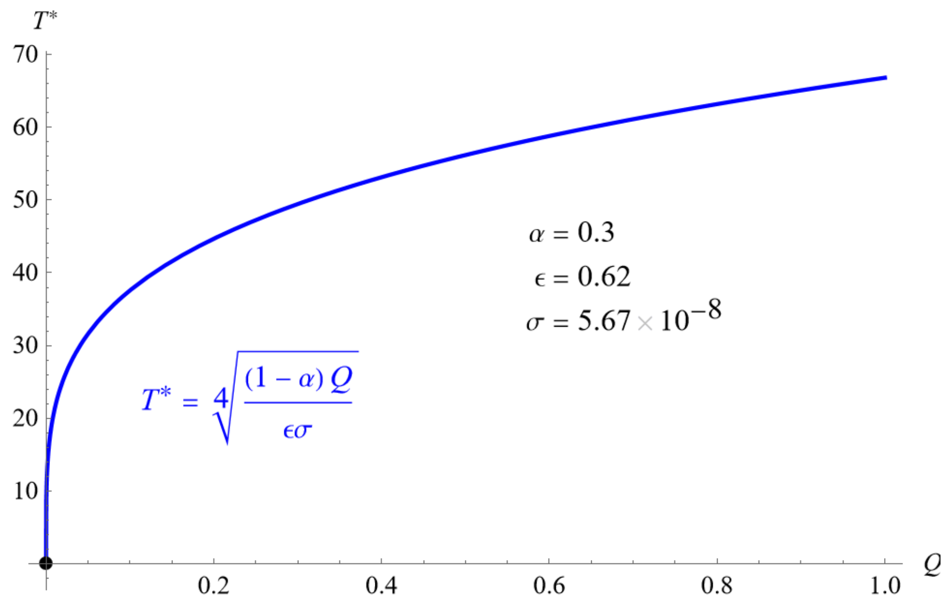
Below is a plot of $\dot{T}$ versus T. Wherever the graph is above the horizontal axis, the flow moves to the right, and wherever the graph is below the horizontal axis, the flow moves to the left.



This makes T^* a stable equilibrium point; it's indicated on the graph by a filled-in circle.

Part c)

Below is a plot of T^* versus Q for $\alpha = 0.3$, $\epsilon = 0.62$, and $\sigma = 5.67 \times 10^{-8}$.



Part d)

If $Q = 342$ W/m^2, $\alpha \approx 0.3$, $\sigma = 5.67 \times 10^{-8}$ W/m^{2}K^4, and $\epsilon \approx 0.62$, then

$$T^* = \sqrt[4]{\frac{(1-0.3)(342)}{(0.62)(5.67 \times 10^{-8})}}\ \text{K} \approx 300\ \text{K}.$$